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Variational Stochastic Procedure for Broken Symmetry Phase and Fermionic Fields

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Stochastic quantization of the $O(N)$ φ^4-field theory is studied in the broken symmetry phase by a variational approach. The resulting integral equation is solved analytically to any order in $1/N$ expansion. This method is adapted to fermionic models. In particular, an analytic fermionic toy model is introduced in $D = 0$ and we argue that the D-dimensional case is directly accessible in this way.

1. INTRODUCTION

The stochastic quantization method, introduced by Parisi and Wu [1], gives quantum field theory as the thermal equilibrium limit of a hypothetical stochastic process with respect to a new variable: the stochastic time. It provides a framework in which most of the usual techniques in quantum field theory can be used, such as perturbation expansions or dimensional regularization. It also enables us to develop specific treatments for the quantization of nonperturbative solutions, for instance, the stochastic quantization of instantons in the sine-Gordon and φ^4 models [2].

On the other hand, Ito and Morita have introduced [3] a stochastic Schwinger–Dyson equation which in the equilibrium limit produces the ordinary Schwinger–Dyson equation. The aim was a new approach to studying the spontaneous breakdown of symmetry of the φ^4, Goldstone, and Nambu–Jona-Lasinio models.

This stochastic process also can be described by a Langevin equation, which is a parabolic differential equation involving a random source function. In this context, we have shown that, for $O(N)$ scalar field theory, a variational solution of this equation can be built recursively by a $1/N$ expansion [4].

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Our goal in this paper is to examine the spontaneous symmetry-breaking phenomenon in the framework of this variational procedure. In Section 2 we recall briefly the essential features of stochastic quantization and we introduce, through the example of free boson theory, the calculation of the two-point Green function in stochastic quantization by using integral kernels. The study of the $O(N)$ φ^4-model with symmetry breaking is done in Section 3; we find again the usual behavior of the standard theory in the large-N limit. Then we introduce the variational approach and give a method for calculating the higher orders in $1/N$. After introducing an analytic fermionic toy model, which we are able to treat using the variational approach, we finally discuss in Section 4 the possibility to extend this variational process to the study of more elaborate fermionic theories.

2. USUAL STOCHASTIC QUANTIZATION

In stochastic quantization a fictitious time t is introduced and the fields of the theory are supposed to satisfy a Langevin equation in this new variable.

For a field φ in a D-dimensional Euclidean space, this equation is

$$\frac{\partial \varphi_i(x, t)}{\partial t} = -\frac{\delta S_E[\phi]}{\delta \phi_i(x, t)} + \theta_i(x, t) \tag{1}$$

where $S_E[\phi]$ is the Euclidean action of the theory and $\theta_i(x, t)$ is a Gaussian white noise field satisfying

$$\begin{cases} \langle \theta_i(x, t) \rangle_\theta = 0 \\ \langle \theta_i(x, t)\theta_j(x', t') \rangle_\theta = 2\delta^D(x - x')\, \delta(t - t')\, \delta_{i'j} \end{cases} \tag{2}$$

The generalization of these formulas is Wick's theorem:

$$\begin{aligned} \left\langle \prod_{i=1}^{2n} \theta(x_i, t_i) \right\rangle_\sigma &= \frac{1}{2^n n!} \sum_{p \in S_{2m}} \prod_{j=1}^{n} \langle \theta(x_{p(2j-1)}, t_{p(2j-1)}) \rangle_\theta \, \langle \theta(x_{p(2j)}, t_{p(2j)}) \rangle_\theta \\ &= \mathrm{Hf}(\langle \theta(x_j, t_j)\theta(x_i, t_i) \rangle_\theta) \end{aligned}$$

where S_n is the permutation group of n elements and $\mathrm{Hf}(M)$ is the Hafnian of the matrix M.

Therefore the stochastic expectation value of a function $F[\theta]$ is given by

$$\langle F[\theta] \rangle_\theta = \int D\theta\, F[\theta] \exp\left[-\frac{1}{4} \int d^D x\, \theta^2(x) \right]$$

Let us study the simple example of the free boson theory. The corresponding action, extended in the fictitious time, is

$$S_E[\phi] = \int d^D x\, dt\, \phi_i(x, t)\Delta_{ij}(x - y)\phi_j(y, t)$$

where $\Delta_{ij}(x - y) = (\Box + m^2)\,\delta^D\,(x - y)\delta_{ij}$.

The general Langevin equation can be equivalently rewritten with the introduction of an integral kernel:

$$\begin{aligned}\frac{\partial\varphi_i(x, t)}{\partial t} &= \int d^P y\, K_{ij}(x, y)\frac{\delta S_E[\phi]}{\delta\phi_j(x, t)} + \theta_i(x, t)\\ &= \int d^P y\, d^P z\, K_{ij}(x, y)\Delta_{jk}(y - z)\phi_k(z) + \theta_i(x, t)\end{aligned} \tag{3}$$

under the condition that the noise correlation is modified as follows:

$$\langle\theta_i(x, t)\theta_j(x', t')\rangle_\theta = 2K_{ij}(x, x')\,\delta(t - t')$$

If we take $K_{i,j}\,(x, y) = [\Delta_{ij}(x - y)]^{-1}$, Equation (3) simplifies to

$$\frac{\partial\varphi_i(x, t)}{\partial t} = -\varphi_i(x, t) + \theta_i(x, t) \tag{4}$$

Its solution is then clearly

$$\frac{\partial\varphi_i(x, t)}{\partial t} = \int_0^{+\infty} dt'\, \exp[-(t - t')]\,\theta_i(x, t') + ce^{-t}$$

The second term of this solution disappears in the equilibrium limit. The correlation functions are readily obtained by using

$$\langle\theta_i(x, t)\theta_j(x', t')\rangle_\theta = 2(\Box + m^2)\delta^D(x - x')\delta(t - t')\delta_{ij}$$

For instance, we have

$$\lim_{t\to\infty}\langle\varphi(x, t)\varphi(x', t')\rangle_\theta = \int \frac{d^D p}{(2\pi)^D}\frac{N}{p^2 + m^2}$$

We finally see that, with this trick, all the specific information about the theory (e.g., its free propagator) is completely contained in the expectation values of the Gaussian noise. We shall use this approach extensively in the rest of the paper.

3. SYMMETRY BREAKING IN THE $O(N)$ φ^4-MODEL

3.1. Simplified Large-N Limit

In this section we extend the variational method discussed in refs. 5 and 6 to models presenting spontaneous symmetry breaking. It is shown that in this large-N limit, the Langevin equation of stochastic quantization offers a direct and simple way to determine the mass gap of the theory.

The Euclidean "stochastic" action of the $O(N)$ φ^4-model reads

$$S[\varphi] = \int d^D x\, dt \left[\frac{1}{2} \partial_\mu \phi(x, t)\, \partial_\mu \phi(x, t) + \frac{1}{2} m^2 \phi^2(x, t) + \frac{\lambda}{4!N} \phi^4(x, t) \right] \quad (5)$$

where

$$\phi^2(x, t) = \sum_{i=1}^{N} \phi_i(x, t)\phi_i(x, t)$$

The Langevin equation is

$$\frac{\partial \varphi_i(x, t)}{\partial t} = -(-\Box + m^2)\phi_i(x, t) - \frac{\lambda}{3!N} \phi^2(x, t)\phi_i(x, t) + \theta_i(x, t) \quad (6)$$

Let us rewrite the field ϕ_i in the form

$$\phi_i(x, t) = \alpha \sqrt{N}\, \delta_{i,N} + \tilde{\phi}_i(x, t) \quad (7)$$

where α is a constant.

Without loss of generality, we can make the hypothesis that the possibility of symmetry breaking occurs only on the direction of the Nth component. Using (3.3) in the Langevin equation (3.2), we obtain

$$\begin{aligned} \frac{\partial \tilde{\varphi}_i(x, t)}{\partial t} &= -(-\Box + m^2)\, \tilde{\phi}_i(x, t) \\ &\quad - \frac{\lambda}{3!N} (\alpha^2 N + \phi^2(x, t) + 2\alpha \sqrt{N \tilde{\phi}_N(x, t)} + \tilde{\phi}_i(x, t)) \\ &\quad - \left(m^2 + \frac{\lambda}{3!N} (\alpha^2 N + \tilde{\phi}^2 + 2\alpha \sqrt{N}) \right) \alpha \sqrt{N}\, \delta_{iN} + \theta_i(x, t) \end{aligned} \quad (8)$$

If we are looking for the strict $N \to \infty$ limit, we can use the factorization property of Migdal [7] and Witten [8]:

$$\lim_{N \to \infty} \phi^2(x, t) = \langle \phi^2(x, t) \rangle_\theta = \sigma(t) \quad (9)$$

Therefore, in this large-N limit and at equilibrium, we find

$$\frac{\partial\tilde{\varphi}_i(x,t)}{\partial t} = -\left[-\Box + m^2 + \frac{\lambda}{3!N}(\alpha_0^2 N + \sigma_0 + 2\alpha_0\sqrt{N}\,\tilde{\phi}_N(x,t))\right]\tilde{\phi}_i(x,t)$$
$$-\left[m^2 + \frac{\lambda}{3!N}(\alpha_0^2 N + \sigma_0 + 2\sqrt{N}\,\alpha_0\tilde{\phi}_N(x,t))\right]\alpha_0\sqrt{N}\delta_{iN}$$
$$+\ \theta_i(x,t) \tag{10}$$

where

$$\begin{cases}\alpha_0 = \lim\limits_{N\to\infty}\alpha \\ \alpha_0 = \lim\limits_{t\to\infty}\sigma(t)\end{cases} \tag{11}$$

Two possibilities occur:

· If $\alpha_0 = 0$, equation (3.6) gives

$$\lim_{N\to\infty}\langle\tilde{\phi}_i(x\tilde{\phi}_i(x)\rangle = \sigma_0 = \int \frac{d^Dp}{(2\pi)^D}\frac{N}{p^2 + m^2 + (\lambda/3!N)\sigma_0} \tag{12}$$

with

$$\Sigma_0 = m^2 + \frac{\lambda}{3!N}\sigma_0 \tag{13}$$

Combining (3.8) and (3.9), we then have

$$\Sigma_0 = m^2 + \frac{\lambda}{3!N}\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2 + \Sigma_0} \tag{14}$$

This is the well-known gap equation for the model under study.

If $\alpha_0 \neq 0$ and $m^2 + (\lambda/3!N)(\alpha_0^2 N + \sigma_0 + 2\alpha_0\sqrt{N}\tilde{\phi}_N(x,t))$, equation (3.6) becomes

$$\frac{\partial\varphi_i(x,t)}{\partial t} = \tilde{\phi}_i(x,t) + \theta_i(x,t)$$

Then $\Sigma_0 = 0$.

We can remark that, since

$$\alpha_0^2 + \frac{2}{\sqrt{N}}\tilde{\phi}_N(0)\alpha_0 + \frac{3!}{\lambda}m^2 + \frac{\sigma_0}{N} = 0$$

$$\text{i.e., } \alpha_0 = -\frac{2}{\sqrt{N}}\phi_N(0) \pm \sqrt{\frac{\tilde{\phi}_N^2(0)}{N^2} - \left[\left(\frac{3!}{\lambda}\right)m^2 + \frac{\sigma_0}{N}\right]}$$

in the case where $\tilde{\phi}_N(0) = 0$, we find

$$\alpha_0^2 = -\left(\frac{3!}{\lambda}m^2 + \int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2}\right)$$

At the initial point we have $\alpha_0^2 = -(3!/\lambda)(m^2 - m_c^2)$ where

$$m_c^2 = -\frac{3!}{\lambda}\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2}$$

This result is consistent with the well known Mermin–Wagner theorem, which, in $D \leq 2$, forbids a phase transition with continuous symmetry breaking [9].

It is important to note that in this large-N limit, the internal and external symmetries are broken at the same critical point

3.2. Variational Approach

Equation (2.1) can be rewritten in integral form as

$$\phi_i(x, t) = \int d^D x\, dt\, G_{m^2}(x - x'; t - t')\left[\eta_i(x, t) - \lambda\cdot\frac{\delta S_E^{\text{int}}[\phi]}{\delta\phi_i(x', t')}\right]$$

Let $\phi_i^{[\alpha,\sigma]}(x, t)$ be a trial field depending upon a set of parameters $\{\alpha, \sigma\}$, where α is a constant parameter and $\sigma(q)$ a functional parameter. Then, minimizing

$$\begin{aligned} V[\alpha, \sigma] &= \lim_{t\to\infty}\Big\langle \text{Tr}\Big[\phi_i^{[\alpha,\sigma]}(x, t) - \int d^D x\, dt\, G_{m^2}(x - x'; t - t') \\ &\qquad \times \left(\theta_i(x', t') - \lambda\frac{\delta S_E^{\text{int}}[\phi]}{\delta\phi_i(x', t')}\right)\Big]\Big\rangle_\theta \\ &= \lim_{t\to\infty}\langle \text{Tr}[\phi_i^{[\alpha,\sigma]}(x, t)\tilde{\phi}_i(x, t)]^2\rangle_\theta \end{aligned} \tag{15}$$

for all $\{\alpha, \sigma\}$ leads to the best variational answer for a given choice of $\phi_i^{[\alpha,\sigma]}$ [4].

Equation (6) is expressible in momentum space with the introduction of an integral kernel, as in Section 2:

$$\begin{aligned} \frac{\partial\phi_i(p, t)}{\partial t} &= -\phi_i(p, t) - \frac{\lambda}{3!N}\frac{1}{p^2 + m^2} \\ &\quad \times \int \frac{d^D q}{(2\pi)^D}\frac{d^D k}{(2\pi)^D}\,\phi_j(q, t)\phi_j(k, t)\phi_i(p - q + k, t) + \theta_i(p, t) \end{aligned}$$

with

$$\langle \theta_i(p, t)\, \theta_j(p', t') \rangle_\sigma = \frac{\delta^D(p + p')}{p^2 + m^2}\, \delta(t - t')\, \delta_{ij} \tag{16}$$

For studying the symmetry-breaking phenomenon, the variational field will be taken in the following form:

$$\begin{aligned}\phi_i^{[\alpha,\sigma]}(x, t) &= \alpha \delta_{iN} \sqrt{N}\, \delta^D(p) + \int_{-\infty}^{+\infty} dt\, G_\sigma(0; t - t')\, \theta_i(p, t') \\ &= \alpha \delta_{iN} \sqrt{N}\, \delta^D(p) + \phi_i^{[\sigma]}(p, t)\end{aligned}$$

where $G_a(0; t - t') = \theta(t - t') \exp[-a(t - t')]$. We then arrive at

$$\lim_{t\to\infty} \langle \phi_i^{[\alpha,\sigma]}(x, t)\, \phi_i^{[\alpha,\sigma]}(x, t) \rangle_\theta = N \left[\alpha^2 + \int \frac{d^D q}{(2\pi)^D} \frac{1}{p^2 + m^2} \right]$$

and $\tilde{\phi}_i(p, t)$ is written as

$$\begin{aligned}\tilde{\phi}_i(p, t) = \phi_i^{[1]}(p, t) &+ \frac{\lambda}{3! N} \frac{1}{p^2 + m^2} \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \int_0^{+\infty} dt'\, G_1(0; t - t') \\ &\times \phi_j^{[\alpha,\sigma]}(q, t')\, \phi_j^{[\alpha,\sigma]}(k, t')\, \phi_i^{[\alpha,\sigma]}(p - q + k, t')\end{aligned}$$

After calculations, the variational potential is obtained as

$$\begin{aligned}&V[\alpha, \sigma] \\ &= \int \frac{d^D p}{(2\pi)^D} \left[1 - \sigma(p) + \left(1 + \frac{2}{N}\right) \frac{\lambda}{3!} \frac{1}{p^2 + m^2} \right. \\ &\quad \left. \times \left(\alpha^2 + \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)\, \sigma(k)} \right) \right]^2 \\ &\quad \times \{\sigma(p)\, [1 + \sigma(p)]\, (p^2 + m^2)\}^{-1} \\ &\quad + \alpha^2 \left[1 + \frac{\lambda}{3! m^2} \left(\alpha^2 + \left(1 + \frac{2}{N}\right) \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)\sigma(p)} \right) \right]^2 \\ &\quad + \frac{4\alpha^4 \lambda}{3! N m^2} \left(1 - \frac{1}{N}\right) \int \frac{d^D p/(2\pi)^D}{(p^2 + m^2)\sigma(p)} \\ &\quad + \frac{\lambda^2}{(3!)^2} \frac{2}{N} \left(1 + \frac{2}{N}\right) \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + m^2} \\ &\quad \times \left[\int \frac{d^D q}{(2\pi)^D} \frac{3\alpha^2}{(q^2 + m^2)[(p - q)^2 + m^2]\sigma(q)} \right.\end{aligned}$$

$$\times \frac{1}{\sigma(p-q)[1+\sigma(q)+\sigma(p-q)]}$$

$$+ \int \frac{d^Dq}{(2\pi)^D} \frac{d^Dk}{(2\pi)^D} \frac{1}{(q^2+m^2)(k^2+m^2)\sigma(q)\sigma(k)}$$

$$\left.\times \frac{1}{\sigma(p-q+k)\,[(p-q+k)^2+m^2]\,[1+\sigma(q)+\sigma(k)+\sigma(p-q+k)]}\right] \tag{17}$$

• In the large-N limit the preceding result becomes

$$\begin{aligned} \lim_{N\to\infty} V[\alpha, \sigma] = \int \frac{d^Dp}{(2\pi)^D} \Bigg[1 - \sigma_0(p + \frac{\lambda}{3!} \frac{1}{p^2+m^2} \\ \times \left(\alpha^2 + \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\sigma_0(k(k^2+m^2)}\right)\Bigg]^2 \\ \times \{\sigma_0(p)(p^2+m^2)\,[1+\sigma_0(p]\}^{-1} \\ + \alpha_0^2 \left[1 + \frac{\lambda}{3!m^2}\left(\alpha_0^2 + \int \frac{d^Dp}{(2\pi)^D} \frac{1}{\sigma_0(p)(p^2+m^2)}\right)\right]^2 \end{aligned} \tag{18}$$

Taking $\Sigma(p) + p^2 = \sigma(p)\,(p^2+m^2)$, we have

$$\begin{aligned} \lim_{N\to\infty} V[\alpha,\sigma] = \int \frac{d^Dp}{(2\pi)^D} \left[m^2 - \sum\nolimits_0(p) + \frac{\lambda}{3!}\left(\alpha_0^2 + \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2+\Sigma_0(k)}\right)\right]^2 \\ \times \{[p^2 + \sum\nolimits_0(p)](p^2+m^2)[p^2+m^2+\Sigma_0(p)]\}^{-1} \\ + \alpha_0^2 \left[1 + \frac{\lambda}{3!m^2}\left(\alpha_0^2 + \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2+\Sigma_0(p)}\right)\right]^2 \end{aligned} \tag{19}$$

After minimization with respect to the parameters α_0 and σ_0, we obtain in the same way as before:

• If $\alpha_0 = 0$, then

$$\Sigma_0(p) = \Sigma_0 = m^2 + \frac{\lambda}{3!} \int \frac{d^D\,p}{(2\,\pi)^D} \frac{1}{p^2+\Sigma_0}$$

• If $\alpha_0 \neq 0$, then

$$\Sigma_0 = 0 \text{ and } \alpha_0^2 = -\frac{3!m^2}{\lambda} - \int \frac{d^D p}{(2\,\pi)^D} \frac{1}{p^2} = -\frac{3!}{\lambda}(m^2 - m_0^2)$$

3.3. Calculation of Higher Orders in $1/N$

By the same iterative procedure as in ref. 5, we can deduce an integral equation for higher order corrections in $1/N$. Let us put

$$\{a_1 = \alpha, \qquad a_2(q) = \Sigma(q)$$

$$V(a_i) = \sum_{n=0}^{n_{\max}} \left(\frac{1}{N}\right)^n V_n[a_i], \qquad F_n^j[a_i] = \frac{\partial\, V_n[a_i]}{\partial\, a_j(q)}$$

The minimization equation is written as

$$\int \frac{d^D r}{(2\pi)^D} \sum_{l=1}^{2} a_n^l \frac{\partial\, F_0^j[a_i(q)]}{\partial a^l(r)} = -F_n^j[a_0^i(q)]$$

In our case, it corresponds to the following system:

$$\begin{cases} \alpha_n^2 \left.\dfrac{\partial^2 V_0[\alpha,\Sigma]}{\partial\alpha^2\,\partial\alpha^2}\right|_{\substack{\alpha=\alpha_0\\ \Sigma=\Sigma_0}} + \displaystyle\int \frac{d^D r}{(2\pi)^D}\,\Sigma_n(r) \left.\frac{\partial^2 V_0[\alpha,\Sigma]}{\partial\Sigma(r)\,\partial\,\alpha^2}\right|_{\substack{\alpha=\alpha_0\\ \Sigma=\Sigma_0}} = -F_n^1(q) \\ \alpha_n^2 \left.\dfrac{\partial^2 V_0[\alpha,\Sigma]}{\partial\alpha^2\,\partial\Sigma(q)}\right|_{\substack{\alpha=\alpha_0\\ \Sigma=\Sigma_0}} + \displaystyle\int \frac{d^D r}{(2\pi)^D}\,\Sigma_n(r) \left.\frac{\partial^2 V_0[\alpha,\Sigma]}{\partial\Sigma(r)\,\partial\,\Sigma(q)}\right|_{\substack{\alpha=\alpha_0\\ \Sigma=\Sigma_0}} = -F_n^2(q) \end{cases}$$

In the following, we will be only interested in the asymmetric phase, the symmetric phase having been already studied in ref. 5. We then have

$$\alpha_0^2 = -\frac{3!}{\lambda} m^2 - \int \frac{d^D r}{(2\,\pi)^D} \frac{1}{p^2}, \qquad \Sigma_0 = 0$$

After calculations, the preceding system takes the form

$$\begin{cases} A\alpha_n^2 - \displaystyle\int \frac{d^D r}{(2\pi)^D}\,\Sigma_n(r) C(r) = F \\ -C\alpha_n^2 + \left(\dfrac{3!}{\lambda}\right)^2 \Sigma_n(q) D(q) + \displaystyle\int \frac{d^D r}{(2\pi)^D}\,\Sigma_n(r) E(q,r) = G(q) \end{cases}$$

where F and $G(q)$ are variable functions of $F_n^i(q)$, and

$$D(q) = \frac{1}{q^2(q^2 + m^2)}; \qquad D = \int \frac{d^D r}{(2\pi)^D} D(q)$$

$$A = D + \frac{\alpha_0^2}{m^4}; \qquad C(q) = \frac{3!}{\lambda} D(q) + \frac{A}{q^4}$$

$$E(q, t) = \frac{3!}{\lambda}\frac{D(q)}{r^4} + \frac{C(r)}{q^4}$$

Finally, we obtain

$$\alpha_n^2 = \frac{1}{A}\left[\int \frac{d^D r}{(2\pi)^D}\, \Sigma_n(r) C(q) + F\right]$$

with

$$\Sigma_n(q) - \int \frac{d^D r}{(2\pi)^D}\, \Sigma_n(r)\, \frac{D(r)}{A} = H(q)$$

where $H(q)$ depends on $G(q)$ and F. To solve the last equation, it is sufficient to pose

$$a = \int \frac{d^D r}{(2\pi)^D}\, \Sigma_n(r)\, \frac{D(r)}{A}$$

Then $\Sigma_n(q) = H(q) + a$, where

$$a = \left\{\int \frac{d^D r}{(2\pi)^D}\, H(r)\right\} \Big/ \left\{1 - \int \frac{d^D r}{(2\pi)^D}\, \frac{D(r)}{A}\right\}$$

a is well defined and positive, since

$$A = D + \frac{\alpha_0^2}{m^4} = -\frac{1}{m^4}\left[\int \frac{d^D r}{(2\pi)^D}\, \frac{p^2 + m^2}{(p^2 + m^2)^2} + \frac{3!m^2}{\lambda}\right]$$

is negative.

Hence we have shown that it is possible to determine successively all the coefficients of the expansion of α_n and $\Sigma_n(q)$ up to a given order in $1/N$.

4. TOY FERMIONIC MODEL

The extension of the variational scheme to fermionic theories is not trivial on many points. To see the possibilities of applying this procedure and also its limitations, we restrict ourselves to the study of a zero-dimensional toy model.

4.1. Analytic Toy Model

Let us considerer a fermionic model in $D = 0$ with an interaction term of the $O(N)$ Gross–Neveu type. Its Euclidean action is

$$S[\psi, \overline{\psi}] = m\overline{\psi}\psi - \frac{g^2}{4N}[\overline{\psi}\psi]^2 \tag{20}$$

Introducing an auxiliary field σ, we can integrate over the fermionic fields.

Then the partition function takes the following form:

$$Z = \int_{-\infty}^{+\infty} d\sigma\,(m-\sigma)^N \exp\left(-\frac{N}{g}\sigma^2\right) = (-i)\left(\frac{ig}{2\sqrt{N}}\right)^N H_N\left(-i\frac{m}{g}\sqrt{N}\right) \tag{21}$$

where H_N is the usual Hermite polynomial of degree N.

The correlation function is

$$\frac{1}{N}\langle\overline{\psi}\psi\rangle = \frac{1}{N}\frac{\partial \log Z}{\partial m} = \frac{2i\sqrt{N}}{g}\frac{H_{N-1}(-i(m/g)\sqrt{N})}{H_N(-i(m/g)\sqrt{N})} \tag{22}$$

Note the clear relationship between these results and those obtained for the $O(N)$ φ^4-theory in zero dimension [4], where the Hermite polynomials are replaced by Hermite functions. The fact that in the present case the partition function is expressed no longer as an infinite serie in g but as a finite polynomial in the same variable is a direct consequence of the properties of the Grassmann variables.

4.2. Simplified Large-N Limit in Stochastic Quantization

Starting from the preceding Euclidean action extended in fictitious time

$$S[\psi, \overline{\psi}] = \int dt\left[m\,\overline{\psi}(t)\psi(t) - \frac{g^2}{4N}(\overline{\psi}(t)\psi(t))^2\right] \tag{23}$$

we find the Langevin equations satisfied by the fields ψ and $\overline{\psi}$:

$$\begin{cases} \dfrac{\partial\psi_i(t)}{\partial t} = -m\psi_i(t) + \dfrac{g^2}{4N}(\overline{\psi}_j(t)\psi_j(t))\psi_i(t) + \eta_i(t) \\ \dfrac{\partial\overline{\psi}_i(t)}{\partial t} = -m\overline{\psi}_i(t) + \dfrac{g^2}{4N}(\overline{\psi}_j(t)\psi_j(t))\overline{\psi}_i(t) + \overline{\eta}_i(t) \end{cases} \tag{24}$$

The fluctuating noises $\eta_i(t)$ and $\overline{\eta}_i(t)$ have to be understood as Grassmann variables, with correlations

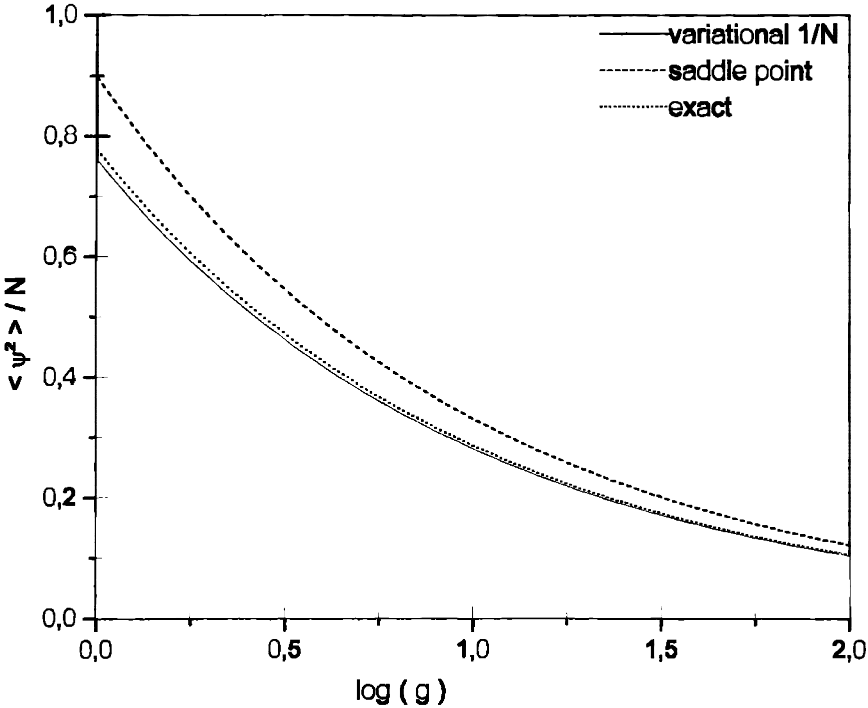


Fig. 1.

$$\langle \eta_i(t)\overline{\eta}_j(t')\rangle_\eta = 2\delta(t - t')\delta_{ij}; \qquad \langle \eta_i\eta_j\rangle = \langle \overline{\eta}_i\overline{\eta}_j\rangle = 0$$

where the spinorial indices have been omitted.

The direct construction of a variational stochastic potential as in the bosonic case is no longer possible. Indeed, the necessary positive-definiteness of the potential is not always ensured because of anticommutativity properties of the trial fields.

To avoid these problems we will introduce an auxiliary field [6], which allows us to lower the degree of nonlinearity of the equation. The action is then written as

$$S_E[\psi, \overline{\psi}, \sigma] = \int dt \left[m\overline{\psi}_i(t)\,\psi_i(t) + \frac{N}{4}\sigma^2(t) - \frac{g}{2}\overline{\psi}_i(t)\psi_i(t)\sigma(t) \right] \quad (25)$$

and the corresponding Langevin equations take the form

$$\begin{cases} \dfrac{\partial \psi_i(t)}{\partial t} = -m\psi_i(t) + \dfrac{g}{2}\sigma(t)\psi_i(t) + \eta_i(t) \\ \dfrac{\partial \overline{\psi}_i(t)}{\partial t} = -m\overline{\psi}_i(t) + \dfrac{g}{2}\sigma(t)\overline{\psi}_i(t) + \overline{\eta}_i(t) \\ \dfrac{\partial \sigma(t)}{\partial t} = -\dfrac{N}{2}\sigma(t) + \dfrac{g}{2}\overline{\psi}_j(t)\psi_j(t) + \theta(t) \end{cases} \tag{26}$$

with $\langle \sigma_i(t)\sigma_j(t')\rangle_\theta = 2\delta(t-t')\delta_{ij}$.

By application of the simplified large-N limit [10] we deduce from the first equation

$$\lim_{\substack{t\to\infty \\ N\to\infty}} \langle \sum_i \psi_i(t)\overline{\psi}_i(t)\rangle = \frac{N}{m - \frac{1}{2}g\sigma_0}$$

where $\lim_{N\to\infty} \sigma(t) = \sigma_0$, and from the third equation,

$$\frac{g}{2}\lim_{\substack{t\to\infty \\ N\to\infty}} \langle \sum_i \psi_i(t)\overline{\psi}_i(t)\rangle = \frac{N}{2}\sigma_0$$

Then $\sigma_0 = -g/(m - \frac{1}{2}g\sigma_0)$. Putting $\Sigma_0 = m - \frac{1}{2}g\sigma_0$, we arrive at the usual gap equation: $\Sigma_0 = m + g^2/(2\,\Sigma_0)$.

4.3. Variational Study

Let us rewrite the auxiliary field as two separate contributions:

$$\sigma(t) = \sigma_s(t) + \sigma_0, \qquad \text{where} \qquad \sigma_0 = \lim_{\substack{t\to\infty \\ N\to\infty}} \langle \sigma(t)\rangle$$

Then, after introduction of "integral" kernels, the Langevin equations become

$$\begin{cases} \dfrac{\partial \psi_i(t)}{\partial t} = -\psi_i(t) + \dfrac{g/2}{m - \frac{1}{2}g\sigma_0}\sigma_s(t)\psi_i(t) + \eta_i(t) \\ \dfrac{\partial \overline{\psi}_i(t)}{\partial t} = -\overline{\psi}_i(t) + \dfrac{g/2}{m - \frac{1}{2}g\sigma_0}\sigma_s(t)\,\overline{\psi}_i(t) + \overline{\eta}_i(t) \\ \dfrac{\partial \sigma(t)}{\partial t} = -\sigma_s(t) - \sigma_0 + \dfrac{g}{2}\overline{\psi}_j(t)\,\psi_j(t) + \theta(t) \end{cases} \tag{27}$$

with

$$\langle \eta_i(t)\overline{\eta}_j(t')\rangle_\eta = \frac{2\delta_{ij}}{m - \frac{1}{2}g\sigma_0}\delta(t-t')$$

$$\langle \theta(t)\theta(t')\rangle_\theta = \frac{4}{N}\delta(t-t')$$

Now, let us construct a variational stochastic potential in the same way as in the bosonic case. For this we choose the variational fields as

$$\psi_i^{[\alpha]}(t) = \int_0^\infty G_\alpha\,(t - t')\,\eta_i(t')\,dt'$$

$$\sigma^{[\beta,\gamma]}(t) = \beta + \int_0^\infty G_\gamma\,(t - t')\,\theta(t')\,dt' = \beta + \sigma_s^{[\gamma]}(t)$$

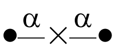
If we note

$$\begin{cases} \tilde{\psi}_i(t) = \displaystyle\int_0^\infty G_1(t - t')\left(\eta_i(t') + \frac{g/2}{m - \frac{1}{2}g\sigma_0}\,\sigma^{[\beta,\gamma]}{}_s(t')\psi_i^{[\alpha]}(t')\right) \\ \tilde{\sigma}(t') = \displaystyle\int_0^\infty G_1\,(t - t')\left[\sigma(t') - \sigma_0\,\delta\,(t) + \frac{g}{N}\,\psi_i^{[\alpha]}(t')\psi^{[\alpha]}{}_i(t')\right] \end{cases}$$

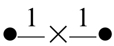
The variation potential is then defined by

$$V[\alpha] = \frac{1}{N}\lim_{t\to\infty}\,[\langle\sum_{i=1}^{N}(\psi_i^{[\alpha]}(t) - \tilde{\psi}_i(t))\rangle_{\eta,\theta}\,\langle\sum_{j=1}^{N}(\psi_j^{[\alpha]}(t) - \tilde{\psi}_j(t))\rangle_{\eta,\theta]}$$

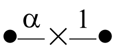
We have to calculate the following graphs:



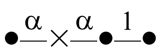
$$= \frac{N}{\alpha(m - \frac{1}{2}g\sigma_0)}$$



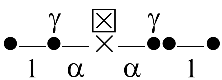
$$= \frac{N}{m - \frac{1}{2}g\sigma_0}$$



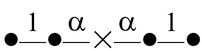
$$= \frac{2N}{(1 + \alpha)(m - \frac{1}{2}g\sigma_0)}$$



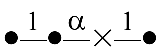
$$= \frac{N}{\alpha(1 + \alpha)(m - \frac{1}{2}g\sigma_0)}$$



$$= \frac{2}{\alpha\gamma(1 + \alpha + \gamma)(m - \frac{1}{2}g\sigma_0)}$$



$$= \frac{N}{\alpha(1 + \alpha)(m - \frac{1}{2}g\sigma_0)}$$



$$= \frac{N}{(1 + \alpha)(m - \frac{1}{2}g\sigma_0)}$$

where the solid and dashed lines correspond, respectively, to the ψ and σ propagators. We finally obtain

$$V[\alpha] = \frac{(1-\alpha^2)}{\alpha(1+\alpha)(m-\frac{1}{2}g\sigma_0)} + \frac{g^2/2}{N(1+\alpha+\gamma)\alpha\gamma(m-\frac{1}{2}g\sigma_0)^3} \tag{28}$$

4.4. Determination of the β and γ Parameters

As seen above, the third Langevin equation in the large N and equilibrium limits gives

$$\lim_{\substack{t\to\infty \\ N\to\infty}} \langle\sigma(t)\rangle = -\frac{g}{m-\frac{1}{2}g\sigma_0} = \sigma_0$$

We can assert that

$$\lim_{\substack{t\to\infty \\ N\to\infty}} \langle\sigma(t)\rangle = \lim_{\substack{t\to\infty \\ N\to\infty}} \langle\sigma^{[\beta,\gamma]}(t)\rangle_\theta = \beta$$

which gives

$$\beta = \sigma_0 = -\frac{g}{m-\frac{1}{2}g\sigma_0}$$

In order to determinate the γ parameter, we can follow the same procedure. Knowing that for the order $1/N$ we have

$$\lim_{t\to\infty} \langle\sigma(t)\sigma(t)\rangle^{(1)} = \lim_{t\to\infty} \frac{\delta^2 S}{\delta\sigma(t)\delta\sigma(t)}\bigg|_{\sigma=\sigma_0} = \frac{2}{1-[\frac{1}{2}g/(m-\frac{1}{2}g\sigma_0)^2]}$$

we prescribe now that

$$\langle\sigma(t)\sigma(t)\rangle^{(1)} = \langle\sigma^{[\beta,\gamma]}(t)\sigma^{[\beta,\gamma]}(t)\rangle_\theta^{(1)} = 1/\gamma$$

and then obtain

$$\gamma = \frac{1}{2}\left(1 - \frac{g^2/2}{(m-\frac{1}{2}g\sigma_0)^2}\right)$$

The only constraint we impose is the coincidence between the exact and variational σ propagators at order $1/N$. Indeed, it is sufficient for our purpose if we remember that we are only interested in the way to determine the ψ propagator.

With this assumption the variational potential can be written as

$$V[\alpha] = \frac{(1-\alpha)^2}{\alpha(1+\alpha)(m - \frac{1}{2}g\sigma_0)} + \frac{1}{N}\frac{g^2}{(\frac{3}{2} + \alpha + \sigma_0^2/4)(m - \frac{1}{2}g\sigma_0)^3(1 - \sigma_0^2/2)} \tag{29}$$

In the large-N limit we simply find $\alpha_0 = 1$.

To obtain the next order, we use the formula

$$\alpha_1 = -\frac{F_1}{\partial F_0/\partial\alpha|_{\alpha=\alpha_0=1}}$$

which gives

$$\alpha_1 = \frac{(7/3 - \sigma_0^2/4)}{(5/2 - \sigma_0^2/4)^2(m - \frac{1}{2}g\sigma_0)^2(1 - \sigma_0^2/2)}$$

Finally, we obtain

$$\lim_{t\to\infty}\left\langle\frac{1}{\mathrm{N}}\right\rangle(\bar{\psi}(t)\psi(t))\rangle = \frac{-1}{m - \frac{1}{2}g\sigma_0}\left(1 - \frac{1}{N}\frac{7/2 - \sigma_0^2/4}{(5/2 - \sigma_0^2/4)^2(m - \frac{1}{2}g\sigma_0)\mathrm{c}^2(1 - \sigma_0^2/2)}\right) \tag{30}$$

Figure compares this result to the exact result in the case $N = 3$.

We observe that the variational result is considerably more accurate than the saddle-point approximation. The exact result is between the two latter approximations. We have reason to hope that this property will be conserved for higher dimensional theory, because the dimensionality appears to play only a minor role in the convergence properties of our $1/N$ expansion scheme, at least for low orders [5, 6].

5. CONCLUSION

In this paper, we have extended our variational stochastic approach of $O(N)$ field theories to the case of asymmetric phase. The systematic introduction of integral kernels simplifies considerably the calculation of the $1/N$ corrections of the self-energy for the φ^4-model. We observe that the $1/N$ corrections break the phase exclusion property which appears in the saddle-point approximation.

For fermionic theory, we were led to introduce an auxiliary field as in ref. 10 in order to lower the degree of nonlinearity of the initial Langevin equation. It enabled us to avoid the nonpositive terms in the variational potential. The method was tested on an analytic zero-dimensional toy model, and showed good agreement with exact results for the propagators.

The use of composite fields in order to treat the renormalization problem will be discussed in a forthcoming paper [11]. We will show how the method developed here enables us to obtain new information about the critical properties of the theory in both phases.

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